CHAPTER 10

Functions

Section-A JEE Advanced/ IIT-JEE

A Fill in the Blanks

1. The values of $f(x)=3\sin\left(\sqrt{\frac{\pi^2}{16}-x^2}\right)$ lie in the interval .... *(1983 - 1 Mark)*

2. For the function $f(x)=\frac{x}{1+e^{1/x}},\ x\neq 0$
$= 0 \quad , \ x=0$
the derivative from the right, $f'(0+)=$ ...., and the derivative from the left, $f'(0-)=$ .... *(1983 - 2 Marks)*

3. The domain of the function $f(x)=\sin^{-1}(\log_2\frac{x^2}{2})$ is given by .... *(1984 - 2 Marks)*

4. Let A be a set of n distinct elements. Then the total number of distinct functions from A to A is .... and out of these .... are onto functions. *(1985 - 2 Marks)*

5. If $f(x)=\sin\ln\left(\frac{\sqrt{4-x^2}}{1-x}\right)$, then domain of $f(x)$ is .... and its range is .... *(1985 - 2 Marks)*

6. There are exactly two distinct linear functions, ..., and .... which map $[-1, 1]$ onto $[0, 2]$. *(1989 - 2 Marks)*

7. If f is an even function defined on the interval (-5, 5), then four real values of x satisfying the equation $f(x)=f\left(\frac{x+1}{x+2}\right)$ are ...., ..., ..., and .... *(1996 - 1 Mark)*

8. If $f(x)=\sin^2 x+\sin^2\left(x+\frac{\pi}{3}\right)+\cos x\cos\left(x+\frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right)=1$, then (gof) (x) = ... *(1996 - 2 Marks)*

B True / False

1. If $f(x)=(a-x^n)^{1/n}$ where $a>0$ and n is a positive integer, then $f[f(x)]=x$. *(1983 - 1 Mark)*

2. The function $f(x)=\frac{x^2+4x+30}{x^2-8x+18}$ is not one -to -one. *(1983 - 1 Mark)*

3. If $f_1(x)$ and $f_2(x)$ are defined on domains D_1 and D_2 respectively, then $f_1(x)+f_2(x)$ is defined on $D_1 \cup D_2$. *(1988 - 1 Mark)*

C MCQs with One Correct Answer

1. Let R be the set of real numbers. If $f: R \to R$ is a function defined by $f(x)=x^2$, then f is : *(1979)*
   (a) Injective but not surjective
   (b) Surjective but not injective
   (c) Bijective
   (d) None of these.

2. The entire graphs of the equation $y = x^2 + kx - x + 9$ is strictly above the x-axis if and only if *(1979)*
   (a) $k<7$ (b) $-5<k<7$
   (c) $k>-5$ (d) None of these.

3. Let $f(x)=|x-1|$. Then *(1983 - 1 Mark)*
   (a) $f(x^2)=(f(x))^2$ (b) $f(x+y)=f(x)+f(y)$
   (c) $f(|x|)=|f(x)|$ (d) None of these

4. If x satisfies $|x-1|+|x-2|+|x-3|\geq 6$, then *(1983 - 1 Mark)*
   (a) $0\leq x\leq 4$ (b) $x\leq -2$ or $x\geq 4$
   (c) $x\leq 0$ or $x\geq 4$ (d) None of these

5. If $f(x)=\cos(\ln x)$, then $f(x)f(y)-\frac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]$ has the value *(1983 - 1 Mark)*
   (a) -1 (b) 1/2
   (c) -2 (d) none of these

6. The domain of definition of the function

$y = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$ is *(1983 - 1 Mark)*

(a) $(-3,-2)$ excluding -2.5 (b) $[0, 1]$ excluding 0.5
(c) $[-2, 1)$ excluding 0 (d) none of these

7. Which of the following functions is periodic? *(1983 - 1 Mark)*

(a) $f(x) = x - [x]$ where $[x]$ denotes the largest integer less than or equal to the real number x
(b) $f(x) = \sin\frac{1}{x}$ for $x \neq 0$, $f(0) = 0$
(c) $f(x) = x\cos x$
(d) none of these

8. Let $f(x) = \sin x$ and $g(x) = \ln|x|$. If the ranges of the composition functions fog and gof are R_1 and R_2 respectively, then *(1994 - 2 Marks)*

(a) $R_1 = \{u : -1 \le u < 1\}$, $R_2 = \{v : -\infty < v < 0\}$
(b) $R_1 = \{u : -\infty < u < 0\}$, $R_2 = \{v : -1 \le v \le 0\}$
(c) $R_1 = \{u : -1 < u < 1\}$, $R_2 = \{v : -\infty < v < 0\}$
(d) $R_1 = \{u : -1 \le u \le 1\}$, $R_2 = \{v : -\infty < v \le 0\}$

9. Let $f(x) = (x+1)^2 - 1$, $x \ge -1$. Then the set $\{x : f(x) = f^{-1}(x)\}$ is *(1995)*

(a) $\left\{0, -1, \frac{-3+i\sqrt{3}}{2}, \frac{-3-i\sqrt{3}}{2}\right\}$
(b) $\{0, 1, -1\}$
(c) $\{0, -1\}$
(d) empty

10. The function $f(x) = |px - q| + r|x|$, $x \in (-\infty, \infty)$ where $p > 0, q > 0, r > 0$ assumes its minimum value only on one point if *(1995)*

(a) $p \neq q$ (b) $r \neq q$
(c) $r \neq p$ (d) $p = q = r$

11. Let $f(x)$ be defined for all $x > 0$ and be continuous. Let $f(x)$ satisfy $f\left(\frac{x}{y}\right) = f(x) - f(y)$ for all x, y and $f(e) = 1$. Then *(1995S)*

(a) $f(x)$ is bounded (b) $f\left(\frac{1}{x}\right) \to 0$ as $x \to 0$
(c) $x f(x) \to 1$ as $x \to 0$ (d) $f(x) = \ln x$

12. If the function $f: [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is *(1999 - 2 Marks)*

(a) $\left(\frac{1}{2}\right)^{x(x-1)}$ (b) $\frac{1}{2}(1 + \sqrt{1 + 4\log_2 x})$
(c) $\frac{1}{2}(1 - \sqrt{1 + 4\log_2 x})$ (d) not defined

13. Let $f: R \to R$ be any function. Define $g: R \to R$ by $g(x) = |f(x)|$ for all x. Then g is *(2000S)*

(a) onto if f is onto
(b) one-one if f is one-one
(c) continuous if f is continuous
(d) differentiable if f is differentiable.

14. The domain of definition of the function $f(x)$ given by the equation $2^x + 2^y = 2$ is *(2000S)*

(a) $0 < x \le 1$ (b) $0 \le x \le 1$
(c) $-\infty < x \le 0$ (d) $-\infty < x < 1$

15. Let $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$. Then for all x, $f(g(x))$ is equal to *(2001S)*

(a) x (b) 1 (c) $f(x)$ (d) $g(x)$

16. If $f: [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$ then $f^{-1}(x)$ equals *(2001S)*

(a) $(x + \sqrt{x^2 - 4})/2$ (b) $x/(1 + x^2)$
(c) $\left(x - \sqrt{x^2 - 4}\right)/2$ (d) $1 + \sqrt{x^2 - 4}$

17. The domain of definition of $f(x) = \frac{\log_2(x+3)}{x^2 + 3x + 2}$ is

(a) $R \setminus \{-1, -2\}$ (b) $(-2, \infty)$ *(2001S)*
(c) $R \setminus \{-1, -2, -3\}$ (d) $(-3, \infty) \setminus \{-1, -2\}$

18. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$. Then the number of onto functions from E to F is *(2001S)*

(a) 14 (b) 16 (c) 12 (d) 8

19. Let $f(x) = \frac{\alpha x}{x+1}$, $x \neq -1$. Then, for what value of α is $f(f(x)) = x$? *(2001S)*

(a) $\sqrt{2}$ (b) $-\sqrt{2}$ (c) 1 (d) -1

20. Suppose $f(x) = (x + 1)^2$ for $x \ge -1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ equals *(2002S)*

(a) $-\sqrt{x} - 1, x \ge 0$ (b) $\frac{1}{(x+1)^2}, x > -1$
(c) $\sqrt{x+1}, x \ge -1$ (d) $\sqrt{x} - 1, x \ge 0$

21. Let function $f: R \to R$ be defined by $f(x) = 2x + \sin x$ for $x \in R$, then f is *(2002S)*

(a) one-to-one and onto
(b) one-to-one but NOT onto
(c) onto but NOT one-to-one
(d) neither one-to-one nor onto

22. If $f: [0, \infty) \longrightarrow [0, \infty)$, and $f(x) = \frac{x}{1+x}$ then f is

(a) one-one and onto *(2003S)*
(b) one-one but not onto
(c) onto but not one-one
(d) neither one-one nor onto

23. Domain of definition of the function

$f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$ for real valued x, is ***(2003S)***

(a) $\left[-\frac{1}{4},\frac{1}{2}\right]$ (b) $\left[-\frac{1}{2},\frac{1}{2}\right]$ (c) $\left(-\frac{1}{2},\frac{1}{9}\right)$ (d) $\left[-\frac{1}{4},\frac{1}{4}\right]$

24. Range of the function $f(x)=\frac{x^2+x+2}{x^2+x+1}; x\in R$ is ***(2003S)***

(a) $(1,\infty)$ (b) $(1, 11/7]$ (c) $(1, 7/3]$ (d) $(1, 7/5]$

25. If $f(x)=x^2+2bx+2c^2$ and $g(x)=-x^2-2cx+b^2$ such that $\min f(x)>\max g(x)$, then the relation between b and c, is ***(2003S)***

(a) no real value of b & c (b) $0<c<b\sqrt{2}$
(c) $|c|<|b|\sqrt{2}$ (d) $|c|>|b|\sqrt{2}$

26. If $f(x)=\sin x+\cos x$, $g(x)=x^2-1$, then $g(f(x))$ is invertible in the domain ***(2004S)***

(a) $\left[0,\frac{\pi}{2}\right]$ (b) $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$ (c) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (d) $[0,\pi]$

27. If the functions f(x) and g(x) are defined on $R\to R$ such that

$f(x)=\begin{cases}0, & x\in\text{rational}\\ x, & x\in\text{irrational}\end{cases}$; $g(x)=\begin{cases}0, & x\in\text{irrational}\\ x, & x\in\text{rational}\end{cases}$ then

$(f-g)(x)$ is ***(2005S)***

(a) one-one & onto
(b) neither one-one nor onto
(c) one-one but not onto
(d) onto but not one-one

28. X and Y are two sets and $f: X\to Y$. If $\{f(c)=y; c\subset X, y\subset Y\}$ and $\{f^{-1}(d)=x; d\subset Y, x\subset X\}$, then the true statement is ***(2005S)***

(a) $f(f^{-1}(b))=b$ (b) $f^{-1}(f(a))=a$
(c) $f(f^{-1}(b))=b, b\subset y$ (d) $f^{-1}(f(a))=a, a\subset x$

29. If $F(x)=\left(f\left(\frac{x}{2}\right)\right)^2+\left(g\left(\frac{x}{2}\right)\right)^2$ where $f''(x)=-f(x)$ and $g(x)=f'(x)$ and given that $F(5)=5$, then $F(10)$ is equal to ***(2006 - 3M, –1)***

(a) 5 (b) 10 (c) 0 (d) 15

30. Let $f(x)=\frac{x}{(1+x^n)^{1/n}}$ for $n\geq 2$ and

$g(x)=\underbrace{(fofo...of)}_{f\text{ occurs } n \text{ times}}(x)$. Then $\int x^{n-2}g(x)dx$ equals.

(2007 -3 marks)

(a) $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}}+K$ (b) $\frac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}}+K$

(c) $\frac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}}+K$ (d) $\frac{1}{n+1}(1+nx^n)^{1+\frac{1}{n}}+K$

31. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x)=e^{x^2}+e^{-x^2}$, $g(x)=xe^{x^2}+e^{-x^2}$ and $h(x)=x^2e^{x^2}+e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then ***(2010)***

(a) $a=b$ and $c\neq b$ (b) $a=c$ and $a\neq b$
(c) $a\neq b$ and $c\neq b$ (d) $a=b=c$

32. Let $f(x)=x^2$ and $g(x)=\sin x$ for all $x\in R$. Then the set of all x satisfying (f o g o g o f) (x) = (g o g o f) (x), where (f o g) (x) = $f(g(x))$, is ***(2011)***

(a) $\pm\sqrt{n\pi}, n\in\{0,1,2,....\}$

(b) $\pm\sqrt{n\pi}, n\in\{1,2,....\}$

(c) $\frac{\pi}{2}+2n\pi, n\in\{...-2,-1,0,1,2....\}$

(d) $2n\pi, n\in\{...-2,-1,0,1,2,....\}$

33. The function $f:[0,3]\to[1,29]$, defined by $f(x)=2x^3-15x^2+36x+1$, is ***(2012)***

(a) one–one and onto (b) onto but not one–one
(c) one–one but not onto (d) neither one–one nor onto

D MCQs with One or More than One Correct

1. If $y=f(x)=\frac{x+2}{x-1}$ then ***(1984 - 3 Marks)***

(a) $x=f(y)$
(b) $f(1)=3$
(c) y increases with x for $x<1$
(d) f is a rational function of x

2. Let $g(x)$ be a function defined on $[-1, 1]$. If the area of the equilateral triangle with two of its vertices at (0,0) and $[x, g(x)]$ is $\frac{\sqrt{3}}{4}$, then the function $g(x)$ is ***(1989 - 2 Marks)***

(a) $g(x)=\pm\sqrt{1-x^2}$ (b) $g(x)=\sqrt{1-x^2}$
(c) $g(x)=-\sqrt{1-x^2}$ (d) $g(x)=\sqrt{1+x^2}$

3. If $f(x)=\cos[\pi^2]x+\cos[-\pi^2]x$, where $[x]$ stands for the greatest integer function, then ***(1991 - 2 Marks)***

(a) $f\left(\frac{\pi}{2}\right)=-1$ (b) $f(\pi)=1$

(c) $f(-\pi)=0$ (d) $f\left(\frac{\pi}{4}\right)=1$

4. If $f(x)=3x-5$, then $f^{-1}(x)$ ***(1998 - 2 Marks)***

(a) is given by $\frac{1}{3x-5}$
(b) is given by $\frac{x+5}{3}$
(c) does not exist because f is not one-one
(d) does not exist because f is not onto

5. If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin\sqrt{x})^2$, then (1998 - 2 Marks)
 (a) $f(x) = \sin^2 x$, $g(x) = \sqrt{x}$
 (b) $f(x) = \sin x$, $g(x) = |x|$
 (c) $f(x) = x^2$, $g(x) = \sin\sqrt{x}$
 (d) f and g cannot be determined.

6. Let $f: (0, 1) \to \mathbf{R}$ be defined by $f(x) = \dfrac{b-x}{1-bx}$, where b is a constant such that $0 < b < 1$. Then
 (a) f is not invertible on (0, 1)
 (b) $f \neq f^{-1}$ on (0, 1) and $f'(b) = \dfrac{1}{f'(0)}$
 (c) $f = f^{-1}$ on (0, 1) and $f'(b) = \dfrac{1}{f'(0)}$
 (d) f^{-1} is differentiable (0, 1)

7. Let $f: (-1, 1) \to IR$ be such that $f(\cos 4\theta) = \dfrac{2}{2-\sec^2\theta}$ for $\theta \in \left(0, \dfrac{\pi}{4}\right) \cup \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$. Then the value (s) of $f\left(\dfrac{1}{3}\right)$ is (are)
 (a) $1-\sqrt{\dfrac{3}{2}}$ (b) $1+\sqrt{\dfrac{3}{2}}$ (c) $1-\sqrt{\dfrac{2}{3}}$ (d) $1+\sqrt{\dfrac{2}{3}}$

8. The function $f(x) = 2|x| + |x+2| - ||x+2| - 2|x||$ has a local minimum or a local maximum at $x =$ (JEE Adv. 2013)
 (a) -2 (b) $\dfrac{-2}{3}$ (c) 2 (d) $\dfrac{2}{3}$

9. Let $f: \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to R$ be given by $f(x) = (\log(\sec x + \tan x))^3$. Then (JEE Adv. 2014)
 (a) $f(x)$ is an odd function
 (b) $f(x)$ is one-one function
 (c) $f(x)$ is an onto function
 (d) $f(x)$ is an even function

10. Let $a \in R$ and let $f: R \to R$ be given by $f(x) = x^5 - 5x + a$. Then (JEE Adv. 2014)
 (a) $f(x)$ has three real roots if $a > 4$
 (b) $f(x)$ has only real root if $a > 4$
 (c) $f(x)$ has three real roots if $a < -4$
 (d) $f(x)$ has three real roots if $-4 < a < 4$

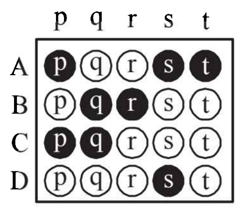
11. Let $f(x) = \sin\left(\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\right)$ for all $x \in R$ and $g(x) = \dfrac{\pi}{2}\sin x$ for all $x \in R$. Let $(fog)(x)$ denote $f(g(x))$ and $(gof)(x)$ denote $g(f(x))$. Then which of the following is (are) true? (JEE Adv. 2015)
 (a) Range of f is $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$
 (b) Range of fog is $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$
 (c) $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = \dfrac{\pi}{6}$
 (d) There is an $x \in R$ such that $(gof)(x) = 1$

E Subjective Problems

1. Find the domain and range of the function $f(x) = \dfrac{x^2}{1+x^2}$. Is the function one-to-one? (1978)
2. Draw the graph of $y = |x|^{1/2}$ for $-1 \le x \le 1$. (1978)
3. If $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$, find $f(6)$. (1979)
4. Consider the following relations in the set of real numbers R.
 $R = \{(x, y); x \in R, y \in R, x^2 + y^2 \le 25\}$
 $R' = \left\{(x, y): x \in R, y \in R, y \ge \dfrac{4}{9}x^2\right\}$
 Find the domain and range of $R \cap R'$. Is the relation $R \cap R'$ a function? (1979)
5. Let A and B be two sets each with a finite number of elements. Assume that there is an injective mapping from A to B and that there is an injective mapping from B to A. Prove that there is a bijective mapping from A to B. (1981 - 2 Marks)
6. Let f be a one-one function with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$. It is given that exactly one of the following statements is true and the remaining two are false $f(x) = 1, f(y) \neq 1, f(z) \neq 2$ determine $f^{-1}(1)$. (1982 - 3 Marks)
7. Let R be the set of real numbers and $f: R \longrightarrow R$ be such that for all x and y in R $|f(x) - f(y)| \le |x-y|^3$. Prove that $f(x)$ is a constant. (1988 - 2 Marks)
8. Find the natural number 'a' for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function '$f$' satisfies the relation $f(x+y) = f(x)f(y)$ for all natural numbers x, y and further $f(1) = 2$. (1992 - 6 Marks)
9. Let $\{x\}$ and $[x]$ denotes the fractional and integral part of a real number x respectively. Solve $4\{x\} = x + [x]$. (1994 - 4 Marks)
10. A function $f: IR \to IR$, where IR is the set of real numbers, is defined by $f(x) = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$. Find the interval of values of α for which f is onto. Is the function one-to-one for $\alpha = 3$? Justify your answer. (1996 - 5 Marks)
11. Let $f(x) = Ax^2 + Bx + C$ where A, B, C are real numbers. Prove that if $f(x)$ is an integer whenever x is an integer, then the numbers $2A$, $A + B$ and C are all integers. Conversely, prove that if the numbers $2A$, $A+B$ and C are all integers then $f(x)$ is an integer whenever x is an integer. (1998 - 8 Marks)

F Match the Following

Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :
If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



1. Let the function defined in column 1 have domain $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and range $(-\infty,\infty)$ ***(1992 - 2 Marks)***

Column I	**Column II**
(A) $1+2x$	(p) onto but not one-one
(B) $\tan x$	(q) one- one but not onto
	(r) one- one and onto
	(s) neither one-one nor onto

2. Let $f(x)=\dfrac{x^2-6x+5}{x^2-5x+6}$ ***(2007 -6 marks)***

Match of expressions/statements in Column I with expressions/statements in Column II and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given in the ORS.

Column I	***Column II***
(A) If $-1<x<1$, then $f(x)$ satisfies	(p) $0<f(x)<1$
(B) If $1<x<2$, then $f(x)$ satisfies	(q) $f(x)<0$
(C) If $3<x<5$, then $f(x)$ satisfies	(r) $f(x)>0$
(D) If $x>5$, then $f(x)$ satisfies	(s) $f(x)<1$

I Integer Value Correct Type

1. Let $f:[0,4\pi]\to[0,\pi]$ be defined by $f(x)=\cos^{-1}(\cos x)$. The number of points $x\in[0,4\pi]$ satisfying the equation

$f(x)=\dfrac{10-x}{10}$ is ***(JEE Adv. 2014)***

Section-B JEE Main / AIEEE

1. The domain of $\sin^{-1}[\log_3(x/3)]$ is **[2002]**
(a) $[1,9]$ (b) $[-1,9]$ (c) $[-9,1]$ (d) $[-9,-1]$

2. The function $f(x)=\log\left(x+\sqrt{x^2+1}\right)$, is **[2003]**
(a) neither an even nor an odd function
(b) an even function
(c) an odd function
(d) a periodic function.

3. Domain of definition of the function $f(x)=\dfrac{3}{4-x^2}+\log_{10}(x^3-x)$, is **[2003]**
(a) $(-1,0)\cup(1,2)\cup(2,\infty)$ (b) $(a,2)$
(c) $(-1,0)\cup(a,2)$ (d) $(1,2)\cup(2,\infty)$.

4. If $f:R\to R$ satisfies $f(x+y)=f(x)+f(y)$, for all $x, y\in R$ and $f(1)=7$, then $\sum_{r=1}^{n} f(r)$ is **[2003]**
(a) $\dfrac{7n(n+1)}{2}$ (b) $\dfrac{7n}{2}$
(c) $\dfrac{7(n+1)}{2}$ (d) $7n+(n+1)$.

5. A function f from the set of natural numbers to integers defined by **[2003]**

$$f(n)=\begin{cases}\dfrac{n-1}{2}, \text{when n is odd}\\ -\dfrac{n}{2}, \text{when n is even}\end{cases}$$ is

(a) neither one -one nor onto
(b) one-one but not onto
(c) onto but not one-one
(d) one-one and onto both.

6. The range of the function $f(x) = {}^{7-x}P_{x-3}$ is [2004]
(a) $\{1,2,3,4,5\}$ (b) $\{1,2,3,4,5,6\}$
(c) $\{1,2,3,4,\}$ (d) $\{1,2,3,\}$

7. If $f: R \to S$, defined by
$f(x) = \sin x - \sqrt{3}\cos x + 1$, is onto, then the interval of S is [2004]
(a) $[-1, 3]$ (b) $[-1, 1]$ (c) $[0, 1]$ (d) $[0, 3]$

8. The graph of the function $y = f(x)$ is symmetrical about the line $x = 2$, then [2004]
(a) $f(x) = -f(-x)$ (b) $f(2+x) = f(2-x)$
(c) $f(x) = f(-x)$ (d) $f(x+2) = f(x-2)$

9. The domain of the function $f(x) = \dfrac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is [2004]
(a) $[1,2]$ (b) $[2,3)$ (c) $[1,2]$ (d) $[2,3]$

10. Let $f : (-1, 1) \to B$, be a function defined by $f(x) = \tan^{-1}\dfrac{2x}{1-x^2}$, then f is both one - one and onto when B is the interval [2005]
(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left[0, \frac{\pi}{2}\right)$ (c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

11. A function is matched below against an interval where it is supposed to be increasing. Which of the following pairs is incorrectly matched? [2005]

	Interval	Function
(a)	$(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$
(b)	$[2, \infty)$	$2x^3 - 3x^2 - 12x + 6$
(c)	$\left(-\infty, \frac{1}{3}\right]$	$3x^2 - 2x + 1$
(d)	$(-\infty, -4)$	$x^3 + 6x^2 + 6$

12. A real valued function $f(x)$ satisfies the functional equation
$$f(x-y) = f(x)f(y) - f(a-x)f(a+y)$$
where a is a given constant and $f(0) = 1$, $f(2a-x)$ is equal to [2005]
(a) $-f(x)$ (b) $f(x)$
(c) $f(a) + f(a-x)$ (d) $f(-x)$

13. The largest interval lying in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ for which the function,
$f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)$, is defined, is [2007]
(a) $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left[0, \frac{\pi}{2}\right)$
(c) $[0, \pi]$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

14. Let $f: N \to Y$ be a function defined as $f(x) = 4x + 3$ where $Y = \{y \in N : y = 4x + 3$ for some $x \in N\}$.
Show that f is invertible and its inverse is [2008]
(a) $g(y) = \dfrac{3y+4}{3}$ (b) $g(y) = 4 + \dfrac{y+3}{4}$
(c) $g(y) = \dfrac{y+3}{4}$ (d) $g(y) = \dfrac{y-3}{4}$

15. Let $f(x) = (x+1)^2 - 1, x \geq -1$
Statement -1 : The set $\{x : f(x) = f^{-1}(x) = \{0, -1\}$
Statement-2 : f is a bijection. [2009]
(a) Statement-1 is true, Statement-2 is true. Statement-2 is not a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is false.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is true. Statement-2 is not a correct explanation for Statement-1.

16. For real x, let $f(x) = x^3 + 5x + 1$, then [2009]
(a) f is onto R but not one-one
(b) f is one-one and onto R
(c) f is neither one-one nor onto R
(d) f is one-one but not onto R

17. The domain of the function $f(x) = \dfrac{1}{\sqrt{|x| - x}}$ is [2011]
(a) $(0, \infty)$ (b) $(-\infty, 0)$
(c) $(-\infty, \infty) - \{0\}$ (d) $(-\infty, \infty)$

10 Functions

Section-A : JEE Advanced/ IIT-JEE

A
1. $\left[0, \frac{3}{\sqrt{2}}\right]$ 2. 0, 1 3. $[-2, -1] \cup [1, 2]$ 4. $n^n, n!$
5. $(-2, 1), [-1, 1]$ 6. $x+1$ and $-x+1$ 7. $\frac{-3 \pm \sqrt{5}}{2}, \frac{3 \pm \sqrt{5}}{2}$ 8. 1

B
1. T 2. T 3. F

C

1. (d)	2. (b)	3. (d)	4. (c)	5. (d)	6. (c)	7. (a)
8. (d)	9. (b)	10. (c)	11. (d)	12. (b)	13. (a)	14. (d)
15. (a)	16. (d)	17. (d)	18. (a)	19. (b)	20. (a)	21. (c)
22. (d)	23. (b)	24. (a)	25. (d)	26. (d)	27. (c)	28. (c)
29. (a)	30. (a)	31. (d)	32. (a)	33. (b)		

D

1. (a, d)	2. (b, c)	3. (a, c)	4. (b)	5. (a)	6. (a, b)	7. (a, b)
8. (a, b)	9. (a, b, c)	10. (b, d)	11. (a, b, c)			

E
1. R, $[0, 1)$; f is not one to one 3. 3

5. domain $= \{x : x \in R, 16x^4 + 81x^2 - 2025 \le 0\}$; range $= \{y : y \in R,\ y \ge \frac{4x^2}{9}\}$; $R \cap R'$ is not a function.

6. y 8. $a = 3$ 9. $-\frac{5}{3}, 0, \frac{5}{3}$ 10. $2 \le \alpha \le 14$, No

F
1. (A) - q; (B) - r 2. (A) -p, r, s ; (B) - q, s ; (C) - q, s ; (D) -p, r, s

I
1. 3

Section-B : JEE Main/ AIEEE

1. (a)	2. (c)	3. (a)	4. (a)	5. (d)	6. (d)	7. (a)
8. (b)	9. (b)	10. (d)	11. (c)	12. (a)	13. (b)	14. (d)
15. (b)	16. (b)	17. (b)				

Section-A JEE Advanced/ IIT-JEE

A. Fill in the Blanks

1. For the given function to be defined

$\frac{\pi^2}{16} - x^2 \ge 0 \Rightarrow -\pi/4 \le x \le \pi/4$

$\therefore D_f = [-\pi/4, \pi/4]$

Now, for $x \in [-\pi/4, \pi/4]$, $\sqrt{\pi^2/16 - x^2} \in [0, \pi/4]$

and sine function increases on $[0, \pi/4]$

$$\therefore 0 = \sin 0 \le \sin\sqrt{\frac{\pi^2}{16} - x^2} \le \sin \pi/4 = 1/\sqrt{2}$$

$$\Rightarrow 0 \le 3\sin\sqrt{\frac{\pi^2}{16} - x^2} \le 3/\sqrt{2}$$

$\therefore f(x) = [0, 3/\sqrt{2}]$

2. $f(x) = \begin{cases} \frac{x}{1+e^{1/x}}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

$$f'(0^+) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h}{1+e^{1/h}}}{h} = \lim_{h \to 0} \frac{1}{1+e^{1/h}}$$

$$= \lim_{h \to 0} \frac{e^{-1/h}}{e^{-1/h} + 1} = \frac{0}{1} = 0$$

$$f'(0^-) = \lim_{h \to 0} \frac{f(0) - f(0-h)}{h}$$

$$= \lim_{h\to 0} \frac{0-\frac{-h}{1+e^{-1/h}}}{h} = \lim_{h\to 0}\frac{1}{1+e^{-1/h}} = 1$$

Thus $f'(0^+) = 0$ and $f'(0^-) = 1$

3. To find domain of function $f(x) = \sin^{-1}\left(\log_2 \frac{x^2}{2}\right)$

For $f(x)$ to be defined we should have, $-1 \le \log_2\left(\frac{x^2}{2}\right) \le 1$

NOTE THIS STEP:

$\Rightarrow \quad 2^{-1} \le \frac{x^2}{2} \le 2^1 \quad \Rightarrow \quad 1 \le x^2 \le 4$

$\Rightarrow -2 \le x \le -1$ or $1 \le x \le 2$

$\Rightarrow x \in [-2, -1] \cup [1, 2]$

4. Set A has n distinct elements.
Then to define a function from A to A, we need to associate each element of set A to any one the n elements of set A. So total number of functions from set A to set A is equal to the number of ways of doing n jobs where each job can be done in n ways. The total number such ways is $n \times n \times n \times .... \times n$ (n - times).
Hence the total number of functions from A to A is n^n.
Now for an onto function from A to A, we need to associate each element of A to one and only one element of A. So total number of onto functions from set A to A is equal to number of ways of arranging n elements in range (set A) keeping n elements fixed in domain (set A). n elements can be arranged in $n!$ ways.
Hence, the total number of functions from A to A is $n!$.

5. The given function is,

$$f(x) = \sin\left[\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right]$$

For ℓn to be defined $\frac{\sqrt{4-x^2}}{1-x} > 0 \Rightarrow 1-x > 0$

Also $4 - x^2 > 0 \Rightarrow x < 1$ and $-2 < x < 2$

Combining these two inequalities, we get $x \in (-2, 1)$

$\therefore$ Domain of f is $(-2, 1)$

Also $\sin\theta$ always lies in $[-1, 1]$.

$\therefore$ Range of f is $[-1, 1]$

6. **KEY CONCEPT :** Every linear function is either strictly increasing or strictly decreasing. If $f(x) = ax + b$, $D_f = [p, q]$, $R_f = [m, n]$
Then $f(p) = m$ and $f(q) = n$, if $f(x)$ is strictly increasing and $f(p) = n, f(q) = m$, If $f(x)$ is strictly decreasing function.
Let $f(x) = ax + b$ be the linear function which maps $[-1, 1]$ onto $[0, 2]$. then $f(-1) = 0$ and $f(1) = 2$
or $f(-1) = 2$ and $f(1) = 0$
Depending upon $f(x)$ is increasing or decreasing respectively.

$\Rightarrow \quad -a + b = 0$ and $a + b = 2$ ....(1)

or $\quad -a + b = 2$ and $a + b = 0$ ....(2)

Solving (1), we get $a = 1, b = 1$.
Solving (2), we get $a = -1, b = 1$
Thus there are only two functions i.e., $x + 1$ and $-x + 1$.

7. Given that $f(x) = f\left(\frac{x+1}{x+2}\right)$ and f is an even function

$\therefore f(x) = f(-x) = f\left(\frac{-x+1}{-x+2}\right)$

$\Rightarrow x = \frac{-x+1}{-x+2} \quad \Rightarrow x^2 - 3x + 1 = 0 \quad \Rightarrow x = \frac{3 \pm \sqrt{5}}{2}$

Also $f(x) = f\left(\frac{x+1}{x+2}\right) = f(-x)$

$\Rightarrow \frac{x+1}{x+2} = -x \Rightarrow x^2 + 3x + 1 = 0 \Rightarrow x = \frac{-3 \pm \sqrt{5}}{2}$

$\therefore$ Four values of x are

$\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$

8. $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$

$\Rightarrow f(x) = \sin^2 x + \left[\sin\left(x + \frac{\pi}{3}\right)\right]^2 + \cos x \cos\left(x + \frac{\pi}{3}\right)$

$\Rightarrow f(x) = \sin^2 x + \frac{1}{4}(\sin x + \sqrt{3}\cos x)^2$

$+ \frac{1}{2}\cos x(\cos x - \sqrt{3}\sin x)$

$= \frac{5}{4}(\sin^2 x + \cos^2 x) = \frac{5}{4}$

$\therefore \quad (gof)x = g[f(x)] = g(5/4) = 1$

B. True/False

1. $f(x) = (a - x^n)^{1/n}$, $a > 0$, n is + ve integer
$f(f(x)) = f[(a - x^n)^{1/n}] = [a - \{(a - x^n)^{1/n}\}^n]^{1/n}$
$= (a - a + x^n)^{1/n} = x$

2. **KEY CONCEPT :** A function is one-one if it is strictly increasing or strictly decreasing, other wise it is many one.

$f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18} \Rightarrow \quad f'(x) = \frac{-12[x^2 + 2x - 26]}{(x^2 - 8x + 18)^2}$

$\Rightarrow \quad f'(x) = \frac{-12(x - 3\sqrt{3} + 1)(x + 3\sqrt{3} + 1)}{(x^2 - 8x + 18)^2}$

$\Rightarrow \quad f(x)$ increases on $\left(-3\sqrt{3} - 1, 3\sqrt{3} - 1\right)$ and decreases otherwise.

$\Rightarrow \quad f(x)$ is many one.

3. We know that sum of any two functions is defined only on the points where both f_1 as well as f_2 are defined that is $f_1 + f_2$ is defined on $D_1 \cap D_2$.
$\therefore$ The given statement is false.

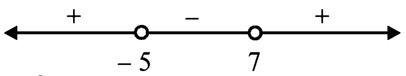
C. MCQs with ONE Correct Answer

1. **(d)** $f(x) = x^2$ is many one as $f(1) = f(-1) = 1$
Also f is into as – ve real number have no pre-image.
$\therefore$ F is neither injective nor surjective.

2. **(b)** $y = x^2 + (k-1)x + 9 = \left(x + \frac{k+1}{2}\right)^2 + 9 - \left(\frac{k-1}{2}\right)^2$

For entire graph to be above x-axis, we should have

$$9 - \left(\frac{k-1}{2}\right)^2 > 0$$



$\Rightarrow \; k^2 - 2k - 35 < 0 \; \Rightarrow \; (k-7)(k+5) < 0$

i.e, $-5 < k < 7$

3. **(d)** $f(x) = |x-1| = \begin{cases} -x+1, & x < 1 \\ x-1, & x \geq 1 \end{cases}$

Consider $f(x^2) = (f(x))^2$

If it is true it should be $\forall \; x$

$\therefore$ Put $x = 2$

LHS $= f(2^2) = |4-1| = 3$ and RHS $= (f(2))^2 = 1$

$\therefore$ (a) is not correct

Consider $f(x+y) = f(x) + f(y)$

Put $x = 2, y = 5$ we get

$f(7) = 6; f(2) + f(5) = 1 + 4 = 5$

$\therefore$ (b) is not correct

Consider $f(|x|) = |f(x)|$

Put $x = -5$ then $f(|-5|) = f(5) = 4$

$|f(-5)| = |-5-1| = 6$

$\therefore$ (c) is not correct.

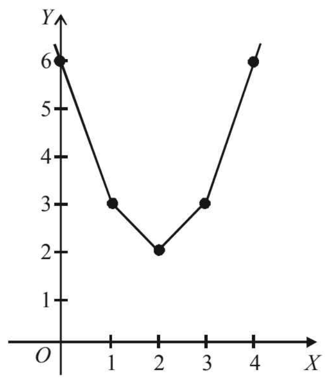
Hence (d) is the correct alternative.

4. **(c)** $|x-1| + |x-2| + |x-3|) \geq 6$

Consider $f(x) = |x-1| + |x-2| + |x-3|$

$$f(x) = \begin{cases} 6-3x, & x < 1 \\ 4-x, & 1 \leq x < 2 \\ x, & 2 \leq x < 3 \\ 3x-6, & x \geq 3 \end{cases}$$

NOTE THIS STEP :



Graph of $f(x)$ shows $f(x) \geq 6$ for $x \leq 0$ or $x \geq 4$

5. **(d)** $f(x) = \cos(\log x)$

$$\therefore \quad f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$$

$= \cos(\log x)\cos(\log y) - \frac{1}{2}[\cos(\log x - \log y) + \cos(\log x + \log y)]$

$= \cos(\log x)\cos(\log y) - \frac{1}{2}[2\cos(\log x)\cos(\log y)]$

$= 0$

6. **(c)** $y = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$

$y = f(x) + g(x)$

NOTE THIS STEP : Then domain of given function is $D_f \cap D_g$

Now, for domain of $f(x) = \frac{1}{\log_{10}(1-x)}$

We know it is defined only when $1 - x > 0$ and $1 - x \neq 1$

$\Rightarrow x < 1$ and $x \neq 0$

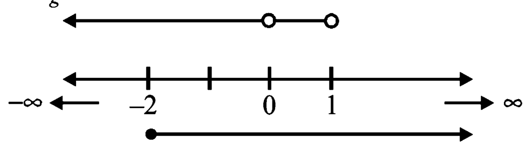
$\therefore D_f = (-\infty, 1) - \{0\}$

For domain of $g(x) = \sqrt{x+2}$

$x + 2 \geq 0$

$\Rightarrow \; x \geq -2$

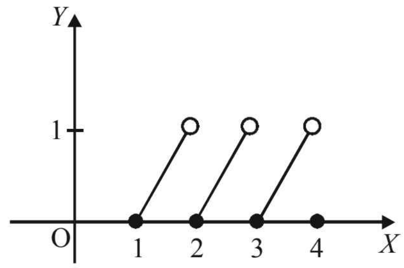
$\therefore D_g = [-2, \infty)$



$\therefore$ Common domain is $[-2, 1) - \{0\}$

7. **(a)** $f(x) = x - [x] = \begin{cases} \cdots \\ x-1, & 1 \leq x < 2 \\ x-2, & 2 \leq x < 3 \\ x-3, & 3 \leq x < 4 \\ \cdots \end{cases}$

graph of function is



Clearly it is a periodic function with period 1.

$\therefore$ (a) is the correct alternative.

8. **(d)** We have $fog(x) = f(g(x)) = \sin(\ln|x|)$

$\therefore \quad R_1 = \{u : -1 \leq u \leq 1\}$

$(\because -1 \leq \sin\theta \leq 1, \forall \theta)$

Also $gof(x) = g(f(x)) = \ln|\sin x|$

$\because \quad 0 \leq |\sin x| \leq 1$

$\therefore \quad -\infty < \ln|\sin x| \leq 0$

$\therefore \quad R_2 = \{v : -\infty < v \leq 0\}$

9. **(c)** $f(x) = f^{-1}(x) \Rightarrow fof(x) = x$

$[(x+1)^2 - 1 + 1]^2 - 1 = x$

$\Rightarrow \quad (x+1)^4 = x + 1$

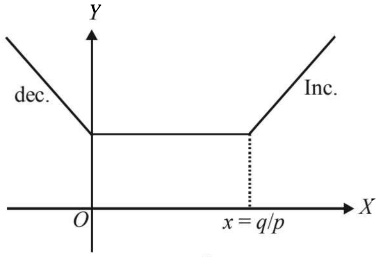
$\therefore \quad x = 0$ or -1

$\therefore \quad$ Req. set is $\{0, -1\}$

10. **(c)** $f(x) = |px - q| + r|x|$

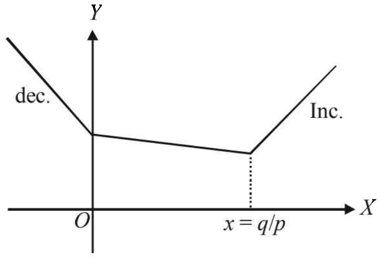
$$= \begin{cases} -px + q - rx, & x \le 0 \\ -px + q + rx, & 0 < x \le q/p \\ px - q + rx, & q/p < x \end{cases}$$

For $r = p, f'(x) < 0$ if $x < 0$
$= 0$ if $0 < x < q/p$
> 0 if $x > q/p$



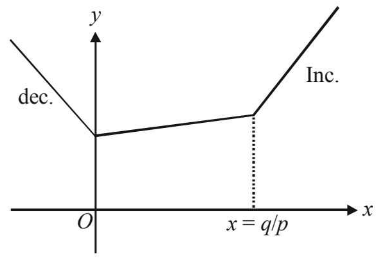
(i)

From graph (i) infinite many points for min value of $f(x)$
for $r < p, f'(x) < 0$ if $x \le 0$
< 0 if $0 < x \le q/p$
> 0 if $x < q/p$



(ii)

From graph (ii) only pt. of min of $f(x)$ at $x = q/p$
For $r > p, f'(x) < 0$ if $x \le 0$
> 0 if $0 < x$



(iii)

From graph (iii) only one pt. of min of $f(x)$ at $x = 0$

11. **(d)** $f(x)$ is continuous and defined for all $x > 0$ and

$$f\left(\frac{x}{y}\right) = f(x) - f(y)$$

Also $f(e) = 1$
$\Rightarrow$ Clearly $f(x) = \ell n\, x$ which satisfies all these properties
$\therefore f(x) = \ell n\, x$

12. **(b)** Let $y = 2^{x(x-1)}$
$\Rightarrow x^2 - x - \log_2 y = 0;$

$$x = \frac{1}{2}\left(1 \pm \sqrt{1 + 4\log_2 y}\right)$$

Since x is + ive, we choose only + out of $\pm$
(for $y \ge 1, \log_2 y \ge 0$)

$$\therefore\ x = \frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 y}\right)$$

or $\quad f^{-1}(x) = \frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$

13. **(c)** Let $h(x) = |x|$ then
$g(x) = |f(x)| = h(f(x))$
Since composition of two continuous functions is continuous, therefore g is continuous if f is continuous.

14. **(d)** It is given that

$2^x + 2^y = 2 \ \forall\ x, y \in R$

but $2^x, 2^y > 0 \ \forall\ x, y \in R$
Therefore, $2^x = 2 - 2^y < 2$
$\Rightarrow\ 0 < 2^x < 2 \Rightarrow x < 1$
Hence domain $= (-\infty, 1)$

15. **(b)** $g(x) = 1 + x - [x];$

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

For integral values of x; $g(x) = 1$
For $x < 0$; (but not integral value) $x - [x] > 0 \Rightarrow g(x) > 1$
For $x > 0$;(but not integral value) $x - [x] > 0 \Rightarrow g(x) > 1$
$\therefore g(x) \ge 1, \forall x \qquad \therefore f(g(x)) = 1, \forall x$

16. **(a)** $f(x) = x + \frac{1}{x} = y \ \Rightarrow x^2 - yx + 1 = 0$

$$\Rightarrow\ x = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

$$\therefore\ x = \frac{y + \sqrt{y^2 - 4}}{2} \qquad (\because\ x \ge 1 \text{ and } y \ge 2)$$

$$\therefore f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$$

NOTE THIS STEP :

17. **(d)** For domain of $f(x) = \frac{\log_2(x+3)}{x^2 + 3x + 2}$
$x^2 + 3x + 2 \ne 0$ and $x + 3 > 0$
$\Rightarrow x \ne -1, -2$ and $x > -3$
$\therefore D_f = (-3, \infty) - \{-1, -2\}$

18. **(a)** From E to F we can define, in all, $2 \times 2 \times 2 \times 2 = 16$ functions (2 options for each element of E) out of which 2 are into, when all the elements of E either map to 1 or to 2.
$\therefore$ No. of onto functions $= 16 - 2 = 14$

19. **(d)** $f(x) = \frac{\alpha x}{x+1}, x \ne -1$

$$f(f(x)) = x \Rightarrow \frac{\alpha\left(\frac{\alpha x}{x+1}\right)}{\frac{\alpha x}{x+1} + 1} = x \Rightarrow \frac{\alpha^2 x}{(\alpha + 1)x + 1} = x$$

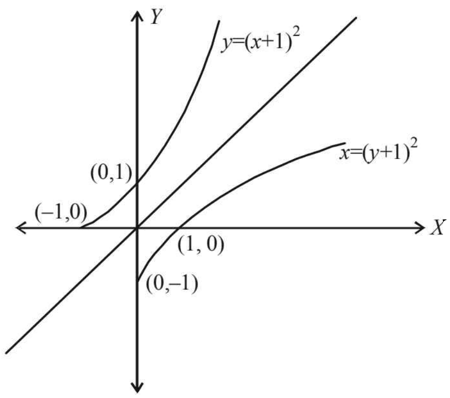
$\Rightarrow (\alpha + 1)x^2 + (1 - \alpha^2)x = 0 \qquad$ ...(1)
$\Rightarrow \alpha + 1 = 0$ and $1 - \alpha^2 = 0 \Rightarrow \alpha = -1$

20. **(d)** Given that $f(x) = (x + 1)^2, x \geq -1$
Now if $g(x)$ is the reflection of $f(x)$ in the line $y = x$ then it can be obtained by interchanging x and y in $f(x)$
i.e., $y = (x + 1)^2$ changes to $x = (y + 1)^2$

$\Rightarrow \quad y+1 = \sqrt{x}$

$\left[\begin{array}{l} y+1 \neq -\sqrt{x}, \text{since } y \geq -1 \\ \text{as in figure.} \end{array}\right.$

$\Rightarrow \quad y = \sqrt{x} - 1 \quad$ defined $\forall\ x \geq 0$



$\therefore g(x) = \sqrt{x} - 1 \quad \forall\ x \geq 0$

21. **(a)** Given that
$f(x) = 2x + \sin x,\ \ x \in R \Rightarrow f'(x) = 2 + \cos x$
But $-1 \leq \cos x \leq 1$
$\Rightarrow 1 \leq 2 + \cos x \leq 3 \Rightarrow 1 \leq 2 + \cos x \leq 3$
$\therefore f'(x) > 0,\ \forall x \in R$
$\Rightarrow f(x)$ is strictly increasing and hence one-one
Also as $x \to \infty$, $f(x) \to \infty$ and $x \to -\infty$, $f(x) \to -\infty$
$\therefore$ Range of $f(x) = R =$ domain of $f(x) \Rightarrow f(x)$ is onto.
Thus, $f(x)$ is one-one and onto.

22. **(b)** Given that $f : [0, \infty) \to [0, \infty)$

Such that $f(x) = \dfrac{x}{x+1}$

Then $f'(x) = \dfrac{1+x-x}{(1+x)^2} = \dfrac{1}{(1+x)^2} > 0\ \forall\ x$

$\therefore f$ is an increasing function $\Rightarrow f$ is one-one.
Also, $D_f = [0, \infty)$

And for range let $\dfrac{x}{1+x} = y \Rightarrow x = \dfrac{y}{1-y}$

$x \geq 0 \Rightarrow 0 \leq y < 1$
$\therefore R_f = [0, 1) \neq$ Co-domain
$\therefore$ f is not onto.

23. **(a)** For $f(x) = \sqrt{\sin^{-1}(2x) + \dfrac{\pi}{6}}$ to be defined and real

if $\sin^{-1} 2x + \pi/6 \geq 0$

$\Rightarrow \sin^{-1} 2x \geq -\dfrac{\pi}{6}$ ...(1)

But we know that
$-\pi/2 \leq \sin^{-1} 2x \leq \pi/2$ ...(2)
Combining (1) and (2), we get

$-\dfrac{\pi}{6} \leq \sin^{-1} 2x \leq \dfrac{\pi}{2}$

$\Rightarrow \sin(-\pi/6) \leq 2x \leq \sin(\pi/2) \Rightarrow -1/2 \leq 2x \leq 1$

$\Rightarrow -1/4 \leq x \leq 1/2 \ \therefore D_f = \left[-\dfrac{1}{4}, \dfrac{1}{2}\right]$

24. **(c)** We have

$f(x) = \dfrac{x^2+x+2}{x^2+x+1} = \dfrac{(x^2+x+1)+1}{x^2+x+1}$

$= 1 + \dfrac{1}{\left(x+\dfrac{1}{2}\right)^2 + \dfrac{3}{4}}$

We can see here that as $x \to \infty$, $f(x) \to 1$ which is the min value of $f(x)$. i.e. $f_{min} = 1$. Also $f(x)$ is max when

$\left(x+\dfrac{1}{2}\right)^2 + \dfrac{3}{4}$ is min which is so when $x = -1/2$

i.e. when $\left(x+\dfrac{1}{2}\right)^2 + \dfrac{3}{4} = \dfrac{3}{4}$.

$\therefore f_{max} = 1 + \dfrac{1}{3/4} = 7/3$

$\therefore R_f = (1, 7/3]$

25. **(d)** We have
$f(x) = x^2 + 2bx + 2c^2$; $g(x) = -x^2 - 2cx + b^2$
$\Rightarrow f(x) = (x + b)^2 + 2c^2 - b^2$
and $g(x) = -(x + c)^2 + b^2 + c^2$
$\Rightarrow f_{min} = 2c^2 - b^2$ and $g_{max} = b^2 + c^2$
For $f_{min} > g_{max} \Rightarrow 2c^2 - b^2 > b^2 + c^2$
$\Rightarrow c^2 > 2b^2 \Rightarrow |c| > |b|\ \sqrt{2}$

26. **(b)** $f(x) = \sin x + \cos x,\ g(x) = x^2 - 1$
$\Rightarrow g(f(x)) = (\sin x + \cos x)^2 - 1 = \sin 2x$

Clearly $g(f(x))$ is invertible in $-\dfrac{\pi}{2} \leq 2x \leq \dfrac{\pi}{2}$

[$\because$ $\sin\theta$ is invertible when $-\pi/2 \leq \theta \leq \pi/2$]

$\Rightarrow -\dfrac{\pi}{4} \leq x \leq \dfrac{\pi}{4}$

27. **(a)** We are given that

$f : R \to R$ such that $f(x) = \begin{cases} 0, & x \in \text{rational} \\ x, & x \in \text{irrational} \end{cases}$

$g : R \to R$ such that $g(x) = \begin{cases} 0, & x \in \text{irrational} \\ x, & x \in \text{rational} \end{cases}$

$\therefore (f - g) : R \to R$ such that

$(f-g)(x) = \begin{cases} -x, \text{ if } & x \in \text{rational} \\ x, \text{ if } & x \in \text{irrational} \end{cases}$

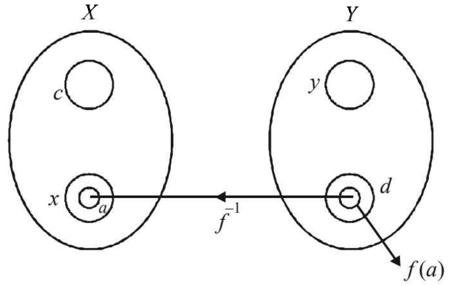
Since $f - g : R \to R$ for any x there is only one value of $(f(x) - g(x))$ whether x is rational or irrational. Moreover as $x \in R$, $f(x) - g(x)$ also belongs to R. Therefore, $(f - g)$ is one-one onto.

28. (d) Given that X and Y are two sets and $f: X \to Y$.

$\{f(c) = y; c \subset X, y \subset Y\}$ and

$\{f^{-1}(d) = x : d \subset Y, x \subset X\}$

The pictorial representation of given information is as shown:



Since $f^{-1}(d) = x \Rightarrow f(x) = d$ Now if $a \subset x$

$\Rightarrow f(a) \subset f(x) = d \Rightarrow \quad f^{-1}[f(a)] = a$

$\therefore \quad f^{-1}(f(a)) \quad = a, a \subset x$ is the correct option.

29. (a) $$F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$$

$$\Rightarrow F'(x) = 2f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right).\frac{1}{2} + 2g\left(\frac{x}{2}\right).g'\left(\frac{x}{2}\right).\frac{1}{2}$$

$$= f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right) + f'\left(\frac{x}{2}\right).f''\left(\frac{x}{2}\right)$$

$[\because g(x) = f'(x) \Rightarrow g'(x) = f''(x)]$

$$= f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right) - f'\left(\frac{x}{2}\right).f\left(\frac{x}{2}\right)$$

$= 0 \qquad [\because f''(x) = -f(x)]$

$\Rightarrow \quad F(x)$ is a constant function.

$\therefore \quad F(x) = F(5) = 5 \ \forall x \in R \Rightarrow \quad F(10) = 5$

30. (a) Given $f(x) = \dfrac{x}{(1+x^n)^{1/n}}$ for $n \geq 2$

$$\therefore \quad fof(x) = f[f(x)] = f\left[\frac{x}{(1+x^n)^{1/n}}\right]$$

$$= \frac{\dfrac{x}{(1+x^n)^{1/n}}}{\left[1 + \dfrac{x^n}{1+x^n}\right]^{1/n}} = \frac{x}{(1+2x^n)^{1/n}}$$

Further, $fofof(x) = \dfrac{x}{(1+3x^n)^{1/n}}$

Proceeding in the similar manner, we get

$$g(x) = fofof \ldots\ldots of(x) = \frac{x}{(1+nx^n)^{1/n}}$$

(f occurs n times)

Now, $\displaystyle\int x^{n-2} g(x)dx = \int \frac{x^{n-1}}{(1+nx^n)^{1/n}}dx$

Let $1 + nx^n = t \quad \Rightarrow \quad n^2 x^{n-1}\, dx = dt$

$$\therefore \text{ Integral becomes } = \frac{1}{n^2}\int t^{-1/n}dt = \frac{1}{n^2}.\frac{t^{-\frac{1}{n}+1}}{-\frac{1}{n}+1} + K$$

$$= \frac{1}{n}.\frac{t^{1-1/n}}{n-1} + K = \frac{(1+nx^n)^{1-1/n}}{n(n-1)} + K$$

31. (d) $f(x) = e^{x^2} + e^{-x^2} \Rightarrow f'(x) = 2x\left(e^{x^2} - e^{-x^2}\right) \geq 0,$

$\forall \text{ x} \in [0,1]$

$\therefore \quad$ f(x) is an increasing function on $[0,1]$

Hence $f_{max} = f(1) = e + \dfrac{1}{e} = a$

$g(x) = xe^{x^2} + e^{-x^2}$

$\Rightarrow \quad g'(x) = (2x^2+1)e^{x^2} - 2xe^{-x^2} \geq 0, \forall x \in [0,1]$

$\therefore \quad g(x)$ is an increasing function on $[0,1]$

$\therefore \quad g_{\max} = g(1) = e + \dfrac{1}{e} = b$

$h(x) = x^2 e^{x^2} + e^{-x^2}$

$h'(x) = 2x\left[e^{x^2}(1+x^2) - e^{-x^2}\right] \geq 0, \forall x \in [0,1]$

$\therefore \quad h(x)$ is an increasing function on $[0,1]$

$\therefore \quad h_{\max} = h(1) = e + \dfrac{1}{e} = c$

Hence $a = b = c$.

32. (a) Given that $f(x) = x^2$ and $g(x) = \sin x, \forall x \in R$

Then (gof) $(x) = \sin x^2$

$\Rightarrow$ (gogof) $(x) = \sin(\sin x^2)$

$\Rightarrow$ (fogogof) $(x) = \sin^2(\sin x^2)$

As given that (fogogof) (x) = (gogof) (x)

$\Rightarrow \sin^2(\sin x^2) = \sin(\sin x^2)$

$\Rightarrow \sin(\sin x^2) = 0, 1$

$\Rightarrow \sin x^2 = n\pi$ or $((4n+1)\dfrac{\pi}{2}$ where $n \in Z$

$\Rightarrow \sin x^2 = 0 \quad \because \sin x^2 \in [-1,1] \Rightarrow x^2 = n\pi$

$\Rightarrow x = \pm\sqrt{n\pi}$ where $n \in W$

33. (b) We have $f(x) = 2x^3 - 15x^2 + 36x + 1$

$\Rightarrow \qquad f'(x) = 6x^2 - 30x + 36$

$= 6(x^2 - 5x + 6)$

$= 6(x-2)(x-3)$

$\because \quad f'(x) > 0 \quad \forall\ x \in [0, 2]$ and $f'(x) < 0 \quad \forall x \in [2, 3]$

$\therefore \quad f(x)$ is increasing on [0, 2] and decreasing on [2, 3]

$\therefore \quad f(x)$ is many one on [0, 3]

Also $f(0) = 1, f(2) = 29, f(3) = 28$

$\therefore$ Global min = 1 and Global max = 29

i.e., Range of f = [1, 29] = codomain

$\therefore f$ is onto.

D. MCQs with ONE or MORE THAN ONE Correct

1. **(a, d)**

Given that $f(x)=y=\dfrac{x+2}{x-1}$

Let us check each option one by one.

(a) $f(x)=\dfrac{x+2}{x-1}=y \Rightarrow x=f(y)$

$\therefore$ (a) is correct

(b) $f(1)\neq 3$ as function is not defined for $x=1$

$\therefore$ (b) is not correct.

(c) $f'(x)=\dfrac{x-1-x-2}{(x-1)^2}=\dfrac{-3}{(x-1)^2}$

$\therefore$ $f'(x)<0$, if $x\neq 1 \Rightarrow f(x)$ is decreasing if $x\neq 1$

$\therefore$ (c) is not correct.

(d) $f(x)=\dfrac{x+2}{x-1}$, which is a rational function of x.

2. **(b, c)**

As (0, 0) and $(x, g(x))$ are two vertices of an equilateral triangle; therefore, length of the side of Δ is

$$=\sqrt{(x-0)^2+(g(x)-0)^2}=\sqrt{x^2+(g(x))^2}$$

$\therefore$ The area of equilateral $\Delta=\dfrac{\sqrt{3}}{4}(x^2+(g(x))^2)$

ATQ, this area $=\dfrac{\sqrt{3}}{4}$

$\therefore$ We get $\dfrac{\sqrt{3}}{4}(x^2+(g(x))^2)=\dfrac{\sqrt{3}}{4}$

$\Rightarrow$ $(g(x))^2=1-x^2 \Rightarrow g(x)=\pm\sqrt{1-x^2}$

$\therefore$ (b), (c) are the correct answers as (a) is not a function ($\because$ image of x is not unique)

3. **(a, c)**

$f(x)=\cos[\pi^2]x+\cos[-\pi^2]x$

We know $9<\pi^2<10$ and $-10<-\pi^2<-9$

$\Rightarrow$ $[\pi^2]=9$ and $[-\pi^2]=-10$

$\Rightarrow$ $\therefore f(x)=\cos 9x+\cos(-10x)$

$f(x)=\cos 9x+\cos 10x$

Let us check each option one by one.

(a) $f\left(\dfrac{\pi}{2}\right)=\cos\dfrac{9\pi}{2}+\cos 5\pi=-1$ **(true)**

(b) $f(\pi)=\cos 9\pi+\cos 10\pi=-1+1=0$ **(false)**

(c) $f(-\pi)=\cos(-9\pi)+\cos(-10\pi)$
$=\cos 9\pi+\cos 10\pi=-1+1=0$ **(true)**

(d) $f\left(\dfrac{\pi}{4}\right)=\cos\dfrac{9\pi}{4}+\cos\dfrac{5\pi}{2}$

$=\cos\left(2\pi+\dfrac{\pi}{4}\right)+0=\dfrac{1}{\sqrt{2}}$ **(false)**

Thus (a) and (c) are the correct options.

4. **(b)** $f(x)=3x-5$ (given), which is strictly increasing on R, so $f^{-1}(x)$ exists.

Let $y=f(x)=3x-5$

$\Rightarrow y+5=3x \Rightarrow x=\dfrac{y+5}{3}$ ...(1)

and $y=f(x) \Rightarrow x=f^{-1}(y)$ ...(2)

From (1) and (2):

$f^{-1}(y)=\dfrac{y+5}{3}=f^{-1}(x)=\dfrac{x+5}{3}$

5. **(a)** Let us check each option one by one.

(a) $f(x)=\sin^2 x$ and $g(x)=\sqrt{x}$

Now, $fog=f(g(x))=f\left(\sqrt{x}\right)=\sin^2\sqrt{x}=(\sin\sqrt{x})^2$

and $gof(x)=g(f(x))=g(\sin^2 x)=\sqrt{\sin^2 x}=|\sin x|$

(a) is true.

(b) $f(x)=\sin x$, $g(x)=|x|$

$fog(x)=f(g(x))=f(|x|)=\sin|x|\neq\left(\sin\sqrt{x}\right)^2$

$\therefore$ (b) is not true

(c) $f(x)=x^2$, $g(x)=\sin\sqrt{x}$

$fog(x)=f[g(x)]=f\left(\sin\sqrt{x}\right)=\left(\sin\sqrt{x}\right)^2$

and $(gof)(x)=g[f(x)]=g(x^2)=\sin\sqrt{x^2}$
$=\sin|x|\neq|\sin x|$

$\therefore$ (c) is not true.

6. **(a, b)** We have $f(x)=\dfrac{b-x}{1-bx}$, $0<b<1$

Let $f(x_1)=f(x_2) \Rightarrow \dfrac{b-x_1}{1-bx_1}=\dfrac{b-x_2}{1-bx_2}$

$\Rightarrow b-b^2x_2-x_1+bx_1x_2=b-x_2-b^2x_1+bx_1x_2$

$\Rightarrow x_2(1-b^2)=x_1(1-b^2)\Rightarrow x_1=x_2$ as $1-b^2\neq 0$

$\therefore f$ is one one.

Also $\dfrac{b-x}{1-bx}=y \Rightarrow b-x=y-bxy \Rightarrow (by-1)x=y-b$

$\Rightarrow x=\dfrac{y-b}{by-1}$

For $y=\dfrac{1}{b}$, x is not defined

$\therefore f$ is neither onto nor invertible.

Also $f'(x)=\dfrac{-1(1-bx)-(-b)(b-x)}{(1-bx)^2}=\dfrac{b^2-1}{(1-bx)^2}$

$\therefore f'(b)=\dfrac{1}{b^2-1}$ and $f'(0)=b^2-1\Rightarrow f'(b)=\dfrac{1}{f'(0)}$

Hence a and b are the correct options.

7. **(a, b)**

Given $f(\cos 4\theta)=\dfrac{2}{2-\sec^2\theta}=\dfrac{2\cos^2\theta}{2\cos^2\theta-1}$

$=\dfrac{1+\cos 2\theta}{\cos 2\theta}=1+\dfrac{1}{\cos 2\theta}$

Let $\cos 4\theta=\dfrac{1}{3} \Rightarrow 2\cos^2 2\theta-1=\dfrac{1}{3} \Rightarrow \cos 2\theta=\pm\sqrt{\dfrac{2}{3}}$

$\therefore f(\cos 4\theta)=1\pm\sqrt{\dfrac{3}{2}}$ or $f\left(\dfrac{1}{3}\right)=1\pm\sqrt{\dfrac{3}{2}}$

8. **(a, b)** For $f(x)= 2|x|+|x+2|-||x+2|-2|x||$

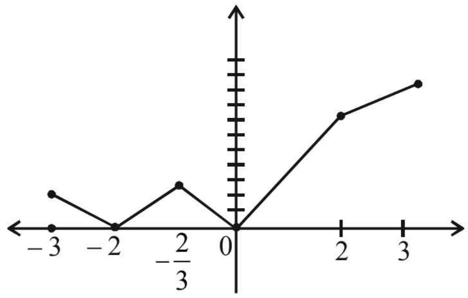
the critical points can be obtained by solving $|x|=0$, $|x+2|=0$ and $||x+2|-2|x||=0$

we get $x=0,-2,2,-\frac{2}{3}$

Then we can write

$$f(x)\begin{cases}-2x-4, & x\le -2\\ 2x+4, & -2<x\le -\frac{2}{3}\\ -4x, & -\frac{2}{3}<x\le 0\\ 4x, & 0<x\le 2\\ 2x+4, & x>2\end{cases}$$

The graph of $y=f(x)$ is as follows



From graph $f(x)$ has local minimum at –2 and 0 and local maximum at $-\frac{2}{3}$

9. **(a, b, c)** $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\longrightarrow R$

$f(x)=[\log(\sec x+\tan x)]^3$

$f(-x)=[\log(\sec x-\tan x)]^3$

$$=\left[\log\left(\frac{(\sec x-\tan x)(\sec x+\tan x)}{\sec x+\tan x}\right)\right]^3$$

$$=\left[\log\left(\frac{1}{\sec x+\tan x}\right)\right]^3=\left[-\log(\sec x+\tan x)\right]^3$$

$$=-\left[\log(\sec x+\tan x)\right]^3=-f(x)$$

$\therefore f$ is an odd function.

(a) is correct and (d) is not correct.

Also

$$f'(x)=3\left[\log(\sec x+\tan x)\right]^2\cdot\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}$$

$$=3\sec x\left[\log(\sec x+\tan x)\right]^2>0\ \forall x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$\therefore f$ is increasing on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

We know that strictly increasing function is one one.

$\therefore f$ is one one

$\therefore$ (b) is correct.

Also $\lim\limits_{x\to\frac{\pi}{2}^-}\left[\log(\sec x+\tan x)\right]^3\to\infty$

and $\lim\limits_{x\to\frac{\pi}{2}^+}\left[\log(\sec x+\tan x)\right]^3\to-\infty$

$\therefore$ Range of $f=(-\infty,\infty)=R$

$\therefore f$ is an onto function.

$\therefore$ (c) is correct.

10. **(b,d)** $f(x)=x^5-5x+a$

$f(x)=0\Rightarrow x^5-5x+a=0\Rightarrow a=5x-x^5=g(x)$

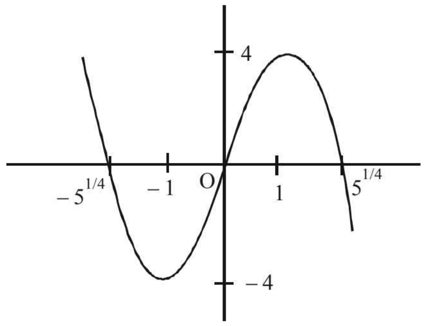
$\Rightarrow g(x)=0$ when $x=0,\ 5^{1/4}\ -5^{1/4}$

and $g'(x)=0\Rightarrow x=1,-1$

Also $g(-1)=-4$ and $g(1)=4$

$\therefore$ graph of $g(x)$ will be as shown below.

From graph



if $a\in(-4,-4)$

then $g(x)=a$ or $f(x)=0$ has 3 real roots

If $a>4$ or $a<-4$

then $f(x)=0$ has only one real root.

$\therefore$ (b) and (d) are the correct options.

11. **(a, b, c)**

$$f(x)=\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$$

$$-1\le\sin x\le 1\Rightarrow -\frac{\pi}{2}\le\frac{\pi}{2}\sin x\le\frac{\pi}{2}$$

$$\Rightarrow\ -1\le\sin\left(\frac{\pi}{2}\sin x\right)\le 1\Rightarrow\frac{-\pi}{6}\le\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\le\frac{\pi}{6}$$

$$\Rightarrow\ \frac{-1}{2}\le\sin\left[\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right]\le\frac{1}{2}$$

$\therefore$ Range of $f=\left[\frac{-1}{2},\frac{1}{2}\right]$

$$fog(x)=\sin\left[\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin x\right)\right)\right]$$

Range of $fog = \left[\frac{-1}{2}, \frac{1}{2}\right]$

$$\lim_{x\to 0} \frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{2}\sin x}$$

$$= \lim_{x\to 0} \frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)} \times \frac{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\pi}{2}\sin x}$$

$= \pi/6$

$$gof(x) = \frac{\pi}{2}\sin\left(\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)\right)$$

$$-\frac{\pi}{2}\sin\left(\frac{1}{2}\right) \le g(f(x)) \le \frac{\pi}{2}\sin\left(\frac{1}{2}\right)$$

$-0.73 \le g(f(x)) \le 0.73$

$\therefore$ $gof(x) \ne 1$ for any $x \in R$.

E. Subjective Problems

1. Since $f(x)$ is defined and real for all real values of x, therefore domain of f is R.

Also $\frac{x^2}{1+x^2} \ge 0$, for all $x \in R$

and $\frac{x^2}{1+x^2} < 1$ $(\because x^2 < 1 + x^2)$ for all $x \in R$

$\therefore$ $0 \le \frac{x^2}{1+x^2} < 1 \Rightarrow 0 \le f(x) < \Rightarrow$ Range of $f = [0, 1)$

Also since $f(1) = f(-1) = 1/2$

$\therefore$ f is not one-to-one.

2. $y = |x|^{1/2}, -1 \le x \le 1$

$\Rightarrow$ $y = \sqrt{-x}$ if $-1 \le x \le 0 = \sqrt{x}$ if $0 \le x \le 1$

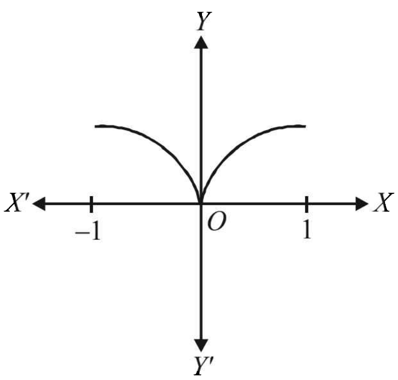
$\Rightarrow$ $y^2 = -x$ if $-1 \le x \le 0$ and $y^2 = x$ if $0 \le x \le 1$

[Here y should be taken always + ve, as by definition y is a + ve square root].

Clearly $y^2 = -x$ represents upper half of left handed parabola (upper half as y is + ve)

and $y^2 = x$ represents upper half of right handed parabola.

Therefore the resulting graph is as shown below :



3. $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$

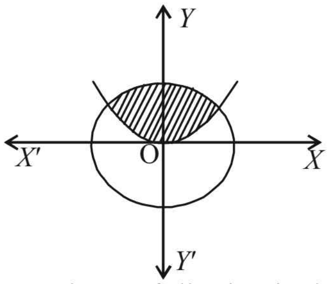
Then $f(6) = 6^9 - 6 \times 6^8 - 2 \times 6^7 + 12 \times 6^6 + 6^4 - 7 \times 6^3 + 6 \times 6^2 + 6 - 3$

$= 6^9 - 6^9 - 2 \times 6^7 + 2 \times 6^7 + 6^4 - 7 \times 6^3 + 6^3 + 6 - 3 = 3$

4. $R = [(x, y); x \in R, y \in R, x^2 + y^2 \le 25]$ which represents all the points inside and on the circle $x^2 + y^2 = 5^2$, with centre (0, 0) and radius = 5

$$R' = \left\{(x, y) : x \in R, y \in R, y \ge \frac{4}{9}x^2\right\}$$

which represents all the points inside and on the upward parabola $x^2 \le \frac{9}{4}y$.



Thus $R \cap R'$ = The set of all points in shaded region.

For domain of $R \cap R'$.

$x^2 + y^2 \le 25$

$\Rightarrow$ $x^2 \le 25 - y^2$ ....(1)

and $y \ge \frac{4}{9}x^2 \Rightarrow \frac{16x^4}{81} \le y^2 \Rightarrow -\frac{16x^4}{81} \ge -y^2$

$\Rightarrow$ $25 - \frac{16x^4}{81} \ge 25 - y^2$ ....(2)

$\therefore$ Combining (1) and (2) $x^2 \le 25 - \frac{16}{81}x^4$

$\Rightarrow$ $16x^4 + 81x^2 - 2025 \le 0$

$\therefore$ Domain of $R \cap R'$ =

$\{x : x \in R, 16x^4 + 81x^2 - 2025 \le 0\}$ and range of $R \cap R'$

$= \{y : y \in R, y \ge \frac{4x^2}{9}, 16x^4 + 81x^2 - 2025 \le 0\}$

$R \cap R'$ is not a function because image of an element is not unique, e..g., (0, 1), (0, 2), (0,3) ..... $\in R \cap R'$.

5. As there is an injective maping from A to B, each element of A has unique image in B. Similarly as there is an injective mapping from B to A, each element of B has unique image in A. So we can conclude that each element of A has unique image in B and each element of B has unique image in A or in other words there is one to one mapping from A to B. Thus there is bijective mapping from A to B.

6. f is one one function,

$D_f = \{x, y, z\}$; $R_f = \{1, 2, 3\}$

Eaxctly one of the following is true :

$f(x) = 1, f(y) \ne 1, f(z) \ne 2$

To determine f^{-1} (1):

Case I: $f(x) = 1$ is true.

$\Rightarrow$ $f(y) \ne 1, f(z) \ne 2$ are false.

$\Rightarrow$ $f(y) = 1, f(z) = 2$ are true.

But $f(x) = 1, f(y) = 1$ are true, is not possible as f is one to one.

$\therefore$ This case is not possible.

Case II: $f(y) \neq 1$ is true.

$\Rightarrow f(x) = 1$ and $f(z) \neq 2$ are false

$\Rightarrow f(x) \neq 1$ and $f(z) = 2$ are true

Thus, $f(x) \neq 1, f(y) \neq 1, f(z) = 2$

$\Rightarrow$ Either $f(x)$ or $f(y) = 2$. So, f is not one to one

$\therefore$ This case is also not possible.

$\therefore f(z) \neq 2$ is true

$\therefore f(x) = 1$ and $f(y) \neq 1$ are false.

$\Rightarrow f(x) \neq 1$ and $f(y) = 1$ are true.

$\Rightarrow f^{-1}(1) = y$

7. Since $|f(x) - f(y)| \leq |x - y|^3$ is true $\forall\ x, y \in R$

We have for $x \neq y$, $\dfrac{|f(x)-f(y)|}{|x-y|} \leq |x-y|^2$

$$\Rightarrow \lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| \leq \lim_{y\to x}|x-y|^2$$

$$\Rightarrow \left|\lim_{y\to x}\frac{f(x)-f(y)}{x-y}\right| \leq 0$$

$\Rightarrow |f'(x)| \leq 0 \Rightarrow f'(x) = 0$

$\Rightarrow f(x)$ is a constant function. Hence Proved.

8. Given that $f(x+y) = f(x)f(y)\ \forall\ x, y \in N$ and $f(1) = 2$

To find 'a' such that,

$$\sum_{k=1}^{n} f(a+k) = 16(2^n - 1) \quad ...(1)$$

For this we start with

$f(1) = 2$ ...(2)

Then $f(2) = f(1+1) = f(1)f(1)$

$\Rightarrow f(2) = 2^2$ Using (2)

Similarly we get, $f(3) = 2^3$,

$f(4) = 2^4, ..., f(n) = 2^n$

Now eq. (1) can be written as

$f(a+1) + f(a+2) + f(a+3) + ... + f(a+n) = 16(2^n - 1)$

$\Rightarrow f(a)f(1) + f(a)f(2) + f(a)f(3) + ... + f(a)f(n)$
$= 16(2^n - 1)$

$\Rightarrow f(a)f(1) + f(a)f(2) + f(a)f(3) + ... + f(a)f(n)$
$= 16(2^n - 1)$

$\Rightarrow f(a)[f(1) + f(2) + f(3) + ... + f(n)] = 16[2^n - 1]$

$\Rightarrow f(a)[2 + 2^2 + 2^3 + ... + 2^n] = 16[2^n - 1]$

$$\Rightarrow f(a)\left[\frac{2(2^n-1)}{2-1}\right] = 16[2^n - 1]$$

$\Rightarrow f(a) = 8 = 2^3 = f(3) \Rightarrow a = 3$

9. Given that $4\{x\} = x + [x]$

Where $[x]$ = greatest integer $\leq x$

$\{x\}$ = fractional part of x

$\therefore x = [x] + \{x\}$ for any $x \in R$

$\therefore$ Given eq^n becomes

$4\{x\} = [x] + \{x\} + [x] \Rightarrow 3\{x\} = 2[x]$

$\Rightarrow [x] = \dfrac{3}{2}\{x\}$ ....(1)

Now $-1 < \{x\} < 1$

$\Rightarrow -\dfrac{3}{2} < \dfrac{3}{2}\{x\} < \dfrac{3}{2}$

$\Rightarrow -\dfrac{3}{2} < [x] < \dfrac{3}{2}$ [Using eqn (1)]

$\Rightarrow [x] = -1, 0, 1$

If $[x] = -1$

$\Rightarrow -1 = \dfrac{3}{2}\{x\}$ [Using eqn (1)]

$\Rightarrow \{x\} = -\dfrac{2}{3}$

$\therefore x = [x] + \{x\}$

$\Rightarrow x = -1 + (-2/3) = -5/3$

If $[x] = 0$

$\Rightarrow \dfrac{3}{2}\{x\} = 0$ [Using eq^n (1)]

$\Rightarrow \{x\} = 0$

$\therefore x = 0 + 0 = 0$

If $[x] = 1$

$\Rightarrow \dfrac{3}{2}\{x\} = 1$ [Using eq^n (1)]

$\Rightarrow \{x\} = 2/3 \Rightarrow x = 1 + 2/3 = 5/3$

Thus, $x = -5/3, 0, 5/3$

10. Let us put $y = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$

$\Rightarrow (\alpha + 6x - 8x^2)y = \alpha x^2 + 6x - 8$

$\Rightarrow (\alpha + 8y)x^2 + 6(1-y)x - (8 + \alpha y) = 0$

Since x is real, $D \geq 0$

$\Rightarrow 36(1-y)^2 + 4(\alpha + 8y)(8 + \alpha y) \geq 0$

$\Rightarrow 9(1 - 2y + y^2) + [8\alpha + (64 + \alpha^2)y + 8\alpha y^2] \geq 0$

$\Rightarrow y^2(9 + 8\alpha) + y(46 + \alpha^2) + (9 + 8\alpha) \geq 0$ ...(1)

For (1) to hold for each $y \in R$, $9 + 8\alpha > 0$

and $(46 + \alpha^2)^2 - 4(9 + 8\alpha)^2 \leq 0 \Rightarrow \alpha > -9/8$

and $[46 + \alpha^2 - 2(9 + 8\alpha)][46 + \alpha^2 + 2(9 + 8\alpha)] \leq 0$

$\Rightarrow \alpha > -9/8$

and $(\alpha^2 - 16\alpha + 28)(\alpha^2 + 16\alpha + 64) \leq 0 \Rightarrow \alpha > -9/8$

and $(\alpha - 2)(\alpha - 14)(\alpha + 8)^2 \leq 0 \Rightarrow \alpha > -8/9$

and $(\alpha - 2)(\alpha - 14) \leq 0$ $[\because (\alpha + 8)^2 \geq 0]$

$\Rightarrow \alpha > -8/9$ and $2 \leq \alpha \leq 14 \Rightarrow 2 \leq \alpha \leq 14$

Thus, $f(x) = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$ will be onto if $2 \leq \alpha \leq 14$.

When $\alpha = 3$

$$f(x) = \frac{3x^2 + 6x - 8}{3 + 6x - 8x^2}$$

In this case $f(x) = 0$ implies, $3x^2 + 6x - 8 = 0$

$$\Rightarrow x = \frac{-6 \pm \sqrt{36+96}}{6} = \frac{-6 \pm \sqrt{132}}{6} = \frac{-6 \pm 2\sqrt{33}}{6}$$

$$= \frac{1}{3}(-3 \pm \sqrt{33})$$

This shows that

$$f\left[\frac{1}{3}(-3+\sqrt{33})\right] = f\left[\frac{1}{3}(-3-\sqrt{33})\right] = 0$$

Therefore, f is not one-to-one at $\alpha = 3$.

11. Suppose $f(x) = Ax^2 + Bx + C$ is an integer whenever x is an integer.

$\therefore$ $f(0), f(1), f(-1)$ are integers

$\Rightarrow$ $C, A+B+C, A-B+C$ are integers.

$\Rightarrow$ $C, A+B, A-B$ are integers

$\Rightarrow$ $C, A+B, (A+B)+(A-B) = 2A$ are integers.

Conversely suppose $2A$, $A+B$ and C are integers.

Let x be any integer.

We have

$$f(x) = Ax^2 + Bx + C$$

$$= 2A\left[\frac{x(x-1)}{2}\right] + (A+B)\,x + C$$

Since x is an integer x, $x(x-1)/2$ is an integer.

Also $2A$, $A+B$ and C are integers.

We get $f(x)$ is an integer for all integer x.

F. Match the Following

1. (A) $f(x) = 1 + 2x, D_f = (-\pi/2, \pi/2)$

The given function represents a straight line so it is one one.

But $f_{min} = 1 - \pi = f\left(-\frac{\pi}{2}\right)$, $f_{max} = 1 + \pi = f\left(\frac{\pi}{2}\right)$

$\therefore$ Range $f = (1-\pi, 1+\pi) \in (-\infty, \infty)$

$\therefore f$ is not onto. Hence (A) $\to$ (q).

(B) $f(x) = \tan x$

It is an increasing function on $(-\pi/2, \pi/2)$ and its range

$= (-\infty, \infty)$ = co-domain of f.

$\therefore f$ is one one onto.

$\therefore$ (B) $\to$ r

2. We have $f(x) = \frac{x^2 - 6x + 5}{x^2 - 5x + 6} = \frac{(x-5)(x-1)}{(x-2)(x-3)}$

(A) If $-1 < x < 1$ then $f(x) = \frac{(-ve)(-ve)}{(-ve)(-ve)} = +ve$

$\therefore$ $f(x) > 0$ (r)

Also $f(x) - 1 = \frac{-x-1}{x^2 - 5x + 6} = -\frac{(x+1)}{(x-2)(x-3)}$

For $-1 < x < 1$, $f(x) - 1 = \frac{-(+ve)}{(-ve)(-ve)} = -ve$

$\Rightarrow f(x) - 1 < 0 \Rightarrow f(x) < 1$ (s)

$\therefore$ $0 < f(x) < 1$ (p)

(B) If $1 < x < 2$ then $f(x) = \frac{(-ve)(+ve)}{(-ve)(-ve)} = -ve$

$\therefore$ $f(x) < 0$ (q) and so $f(x) < 1$ (s)

(C) If $3 < x < 5$ then $f(x) = \frac{(-ve)(+ve)}{(+ve)(+ve)} = -ve$

$\therefore$ $f(x) < 0$ (q) and so $f(x) < 1$ (s)

(D) For $x > 5$, $f(x) > 0$ (r)

Also $f(x) - 1 = \frac{-(x+1)}{(x-2)(x-3)} < 0$

For $x > 5$, $f(x) < 1$ (s)

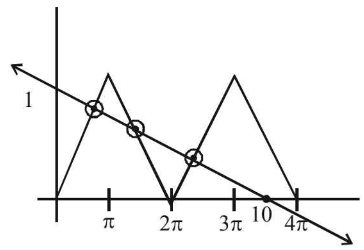
$\therefore$ $0 < f(x) < 1$ (p)

I. Integer Value Correct Type

1. **(3)** We have $f : [0, 4\pi] \to [0, \pi]$

$$f(x) = \cos^{-1}(\cos x)$$

and $g(x) = \frac{10 - x}{10} = 1 - \frac{x}{10}$



The graph of $y = f(x)$ and $y = g(x)$ are as follows.

Clearly $f(x) = g(x)$ has 3 solutions.

Section-B JEE Main/ AIEEE

1. **(a)** $f(x)=\sin^{-1}\left(\log_3\left(\frac{x}{3}\right)\right)$ exists

if $-1 \le \log_3\left(\frac{x}{3}\right) \le 1 \Leftrightarrow 3^{-1} \le \frac{x}{3} \le 3^1$

$\Leftrightarrow 1 \le x \le 9$ or $x \in [1, 9]$

2. **(c)** $f(x)=\log(x+\sqrt{x^2+1})$

$f(-x)=\log\left\{-x+\sqrt{x^2+1}\right\}=\log\left\{\frac{-x^2+x^2+1}{x+\sqrt{x^2+1}}\right\}$

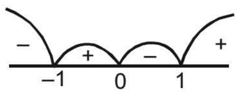
$=-\log(x+\sqrt{x^2+1})=-f(x)$

$\Rightarrow f(x)$ is an odd function.

3. **(a)** $f(x)=\frac{3}{4-x^2}+\log_{10}(x^3-x)$

$4-x^2 \ne 0;\ x^3-x>0;$

$x \ne \pm\sqrt{4}$ and $-1<x<0$ or $1<x<\infty$



$\therefore D=(-1, 0)\cup(1, \infty)-\left\{\sqrt{4}\right\}$

$D=(-1, 0)\cup(1, 2)\cup(2, \infty)$.

4. **(a)** $f(x+y)=f(x)+f(y)$.

Function should be $f(x)=mx$

$f(1)=7;\ \therefore m=7,\ f(x)=7x$

$\sum_{r=1}^{n} f(r)=7\sum_{1}^{n} r=\frac{7n(n+1)}{2}$

5. **(d)** We have $f: N \to I$

If x and y are two even natural numbers,

then $f(x)=f(y)\Rightarrow \frac{-x}{2}=\frac{-y}{2} \Rightarrow x=y$

Again if x and y are two odd natural numbers then

$f(x)=f(y)\Rightarrow \frac{x-1}{2}=\frac{y-1}{2} \Rightarrow x=y$

$\therefore f$ is onto.

Also each negative integer is an image of even natural number and each positive integer is an image of odd natural number.

$\therefore f$ is onto.

Hence f is one one and onto both.

6. **(d)** ${}^{7-x}P_{x-3}$ is defined if

$7-x \ge 0, x-3 \ge 0$ and $7-x \ge x-3$

$\Rightarrow 3 \le x \le 5$ and $x \in \mathbf{I}$

$\therefore x=3, 4, 5$

$\therefore f(3)={}^{7-3}P_{3-3}={}^{4}P_0=1$

$\therefore f(4)={}^{7-4}P_{4-3}={}^{3}P_1=3$

$\therefore f(5)={}^{7-5}P_{5-3}={}^{2}P_2=2$

Hence range $=\{1, 2, 3\}$

7. **(a)** $f(x)$ is onto $\therefore S=$ range of $f(x)$

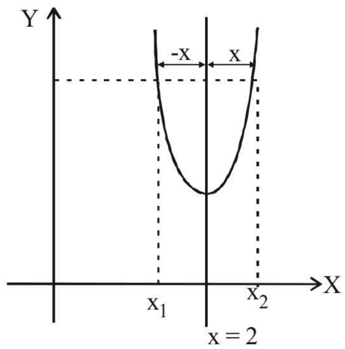
Now $f(x)=\sin x-\sqrt{3}\cos x+1=2\sin\left(x-\frac{\pi}{3}\right)+1$

$\because -1 \le \sin\left(x-\frac{\pi}{3}\right) \le 1$

$-1 \le 2\sin\left(x-\frac{\pi}{3}\right)+1 \le 3$

$\therefore f(x) \in [-1, 3]=S$

8. **(b)** Let us consider a graph symm. with respect to line $x=2$ as shown in the figure.



From the figure

$f(x_1) = f(x_2)$, where $x_1 = 2 - x$ and $x_2 = 2 + x$

$\therefore f(2-x) = f(2+x)$

9. **(b)** $f(x) = \dfrac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is defined

if (i) $-1 \le x-3 \le 1 \Rightarrow 2 \le x \le 4$

and (ii) $9 - x^2 > 0 \Rightarrow -3 < x < 3$

Taking common solution of (i) and (ii),

we get $2 \le x < 3$ $\therefore$ Domain = [2, 3)

10. **(d)** Given $f(x) = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = 2\tan^{-1}x$ for $x \in (-1, 1)$

If $x \in (-1, 1) \Rightarrow \tan^{-1} x \in \left(\dfrac{-\pi}{4}, \dfrac{\pi}{4}\right)$

$\Rightarrow 2\tan^{-1} x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

Clearly, range of $f(x) = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

For f to be onto, codomain = range

$\therefore$ Co-domain of function = $B = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

11. **(c)** Clearly function $f(x) = 3x^2 - 2x + 1$ is increasing when

$f'(x) = 6x - 2 \ge 0 \Rightarrow \quad x \in [1/3, \infty)$

$\therefore f(x)$ is incorrectly matched with $\left(-\infty, \dfrac{1}{3}\right)$

12. **(a)** $f(2a-x) = f(a-(x-a))$

$= f(a)f(x-a) - f(0)f(x) = f(a)f(x-a) - f(x)$

$= -f(x)$

$[\because x = 0, y = 0,\ f(0) = f^2(0) - f^2(a)$

$\Rightarrow f^2(a) = 0 \Rightarrow f(a) = 0]$

$\Rightarrow f(2a-x) = -f(x)$

13. **(b)** $f(x) = 4^{-x^2} + \cos^{-1}\left(\dfrac{x}{2} - 1\right) + \log(\cos x)$

$f(x)$ is defined if $-1 \le \left(\dfrac{x}{2} - 1\right) \le 1$ and $\cos x > 0$

or $\quad 0 \le \dfrac{x}{2} \le 2$ and $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

or $\quad 0 \le x \le 4$ and $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$\therefore \quad x \in \left[0, \dfrac{\pi}{2}\right)$

14. **(d)** Clearly f is one one and onto, so invertible

Also $f(x) = 4x + 3 = y \Rightarrow x = \dfrac{y-3}{4}$

$\therefore \; g(y) = \dfrac{y-3}{4}$

15. **(b)** Given that $f(x) = (x+1)^2 - 1,\ x \ge -1$

Clearly D_f = [–, ∞) but co-domain is not given. Therefore $f(x)$ need not be necessarily onto.

But if $f(x)$ is onto then as $f(x)$ is one one also, $(x+1)$ being something +ve, $f^{-1}(x)$ will exist where

$(x+1)^2 - 1 = y$

$\Rightarrow x + 1 = \sqrt{y+1}$ (+ve square root as $x + 1 \ge 0$)

$\Rightarrow x = -1 + \sqrt{y+1} \Rightarrow f^{-1}(x) = \sqrt{x+1} - 1$

Then $\quad f(x) = f^{-1}(x) \Rightarrow (x+1)^2 - 1 = \sqrt{x+1} - 1$

$\Rightarrow (x+1)^2 = \sqrt{x+1} \Rightarrow (x+1)^4 = (x+1)$

$\Rightarrow (x+1)[(x+1)^3 - 1] = 0 \Rightarrow x = -1, 0$

$\therefore$ The statement-1 is correct but statement-2 is false.

16. **(b)** Given that $f(x) = x^3 + 5x + 1$

$\therefore \quad f'(x) = 3x^2 + 5 > 0, \quad \forall x \in R$

$\Rightarrow \quad f(x)$ is strictly increasing on R

$\Rightarrow \quad f(x)$ is one one

$\therefore \quad$ Being a polynomial $f(x)$ is cont. and inc.

on R with $\lim\limits_{x\to\infty} f(x) = -\infty$

and $\lim\limits_{x\to\infty} f(x) = \infty$

$\therefore \quad$ Range of $f = (-\infty, \infty) = R$

Hence f is onto also. So, f is one one and onto R.

17. **(b)** $f(x) = \dfrac{1}{\sqrt{|x| - x}}$, define if $|x| - x > 0$

$\Rightarrow |x| > x, \Rightarrow x < 0$

Hence domain of $f(x)$ is $(-\infty, 0)$